# Partial $k$-Parallelisms in Finite Projective Spaces

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**Abstract**

In this paper we consider the following question. What is the maximum number of pairwise disjoint $k$-spreads which exist in $\mathrm{PG}(n,q)$? We prove that if $k+1$ divides $n+1$ and $n>k$ then there exist at least two disjoint $k$-spreads in $\mathrm{PG}(n,q)$ and there exist at least $2^{k+1}-1$ pairwise disjoint $k$-spreads in $\mathrm{PG}(n,2)$. We also extend the known results on parallelism in a projective geometry from which the points of a given subspace were removed.

**Keywords:** Grassmannian, lifted MRD codes, parallelism, projective geometry, $q$-analog, spreads, subspace transversal design.

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# 1 Introduction

A $k$-spread in the $n$-dimensional projective space of finite order $q$, namely PG$(n, q)$ is a set $\mathbb{S}$ of $k$-dimensional subspaces (henceforth called $k$-subspaces) in which each point of PG$(n, q)$ is contained in exactly one element of $\mathbb{S}$. A necessary and sufficient condition that a $k$-spread exists in PG$(n, q)$ is that $k + 1$ divides $n + 1$. The size of a $k$-spread in PG$(n, q)$ is $\frac{q^{n+1}-1}{q^{k+1}-1}$. $k$-spreads were extensively studied since they have many application in projective geometry, e.g. [5, 13].

Parallelism is a well known concept in combinatorial designs. A *parallel class* in a block design, is a set of blocks which partition the set of points of the design. Spreads are also a type of combinatorial design on which a parallelism can be defined [11]. A $k$-spread is called a parallel class as it partition the set of all the points of PG$(n, q)$. A $k$-parallelism in PG$(n, q)$ is a partition of the $k$-subspaces of PG$(n, q)$ into pairwise disjoint $k$-spreads. Some 1-parallelisms of PG$(n, q)$ are known for many years. For $q = 2$ and odd $n$ there is an 1-parallelism in PG$(n, 2)$. Such a parallelism was found in the context of Preparata codes and it is known that many such parallelisms exist [1, 2]. For any other power of a prime $q$, if $n = 2^i - 1$, $i \geq 2$, then an 1-parallelism was shown in [3]. In the last forty years no new parameters for 1-parallelisms were shown until recently, when an 1-parallelism in PG(5,3) was proved to exist in [9]. A $k$-parallelism for $k > 1$ was not known until a 2-parallelism in PG$(5, 2)$ was found by [15].

The difficulty to find new parameters for 1-parallelisms and $k$-parallelisms motivates the following question. What is the maximum number of pairwise disjoint $k$-spreads that exist in PG$(n, q)$? Beutelspacher [4] has proved that if $n$ is odd then there exist $q^{2^{\lfloor \log n \rfloor}} + \cdots + q + 1$ pairwise disjoint 1-spreads in PG$(n, q)$. In general we don't have a proof for the following most simple question. Given $q$, $n$, and $k$, such that $k+1$ divides $n+1$ and $n > k$, do there exist two disjoint $k$-spreads in PG$(n, q)$? In this paper we will give a positive answer for this question. Moreover, we will prove that there exist at least $2^{k+1} - 1$ pairwise disjoint $k$-spreads in PG$(n, 2)$ if $k + 1$ divides $n + 1$ and $n > k$.

One of the main tools for our constructions will come from coding theory. It will based on error-correcting codes in the Grassmannian space which are constructed by lifting matrices of error-correcting codes in the rank-metric. This method is well documented, e.g. [7, 8, 16]. The interest in such construction came as result of a new application of such codes in random network coding [12].

The rest of this paper is organized as follows. In Section 2 we will present the two equivalent ways to handle subspaces, in projective geometry and in the Grassmannian. We will explain the method which transfer matrices into subspaces and rank-metric codes into Grassmannian codes. We will present some basic results and connect them into the theory of projective geometry in general and the theory of spreads in particular. Finally, we will define a design called subspace transversal design which will have an important role in our discussion. In Section 3 we present a construction which produces $2^{k+1} - 1$ disjoint $k$-spreads in PG$(2k+1, 2)$. In Section 4 we prove that for a general $q$ there exist at least two disjoint $k$-spreads in PG$(2k + 1, q)$. In Section 5 we present a recursive construction to obtain two disjoint $k$-spread in PG$(n, q)$ if $k + 1$ divides $n+1$ and $n > k$; and $2^{k+1} - 1$ pairwise disjoint $k$-spreads in PG$(n, 2)$ if $k + 1$ divides $n + 1$ and $n > k$. We will also discuss parallelisms in partial sets of PG$(n, q)$ which will be used in the recursive construction. Conclusion and problems for future research are given in Section 6.

# 2 Representation of subspaces, codes, and spreads

The projective geometry $\text{PG}(n,q)$ consists of $\frac{q^{n+1}-1}{q-1}$ points and $\frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}$ lines. The points are represented by a set of nonzero elements from $\mathbb{F}_q^{n+1}$, of maximum size, in which each two elements are linearly independent. Each element $x$ of these $\frac{q^{n+1}-1}{q-1}$ elements represents $q-1$ elements of $\mathbb{F}_q^{n+1}$ which are the multiples of $x$ by the nonzero elements of $\mathbb{F}_q$. A line in $\text{PG}(n,q)$ consists of $q+1$ points. Given two distinct points $x$ and $y$, there is exactly one line which contains these two points. This line contains $x$ and $y$ and the $q-1$ points of the form $\gamma x + y$, where $\gamma \in \mathbb{F}_q \setminus \{0\}$. A point is a 0-subspace in $\text{PG}(n,q)$, a line is a 1-subspace in $\text{PG}(n,q)$, and a $k$-subspace is constructed by taking a $(k-1)$-space $Y$ and a point $x$ not on $Y$ and all points that are constructed by a linear combination of $x$ with any set of points from $Y$.

The *Grassmannian* $\mathcal{G}_q(n,k)$ consists of all the $k$-dimensional subspaces of $\mathbb{F}_q^n$. Clearly, a $k$-dimensional subspace from $\mathcal{G}_q(n,k)$ is a $(k-1)$-subspace of $\text{PG}(n-1,q)$. Extensive research has been done on the Grassmannian in the past few years. The motivation for this research is the application of codes in the Grassmannian for error-correction in random network coding found recently by Koetter and Kschischang [12].

A subset $\mathbb{C}$ of $\mathcal{G}_q(n,k)$ is called an $(n,M,d,k)_q$ *constant dimension code* if it has size $M$ and minimum subspace distance $d$, where the distance function in $\mathcal{G}_q(n,k)$ is defined by

$$d_S(X,Y) \stackrel{\text{def}}{=} 2k - 2\dim\bigl(X \cap Y\bigr),$$

for any two subspaces $X$ and $Y$ in $\mathcal{G}_q(n,k)$.

Two $k$-dimensional subspaces in $\mathcal{G}_q(n,k)$ are called disjoint if their intersection is the null space. A *spread* in $\mathcal{G}_q(n,k)$ is a set $\mathbb{S}$ of pairwise disjoint $k$-dimensional subspaces, such that each nonzero element of $\mathbb{F}_q^n$ is contained in exactly one element of $\mathbb{S}$. Clearly, such a spread is a $(k-1)$-spread in $\text{PG}(n-1,q)$. Hence, a spread in $\mathcal{G}_q(n,k)$ exists if and only if $k$ divides $n$. A set of $M$ pairwise disjoint spreads in $\mathcal{G}_q(n,k)$ is a set of $M$ pairwise disjoint $(k-1)$-spreads in $\text{PG}(n-1,q)$. Henceforth, our discussion will be in terms of $k$-dimensional subspaces of $\mathcal{G}_q(n,k)$ and will be translated into related results in terms of subspaces in projective geometry. The reason is that some of the new developed theory for constant dimension codes will serve as the building blocks for our constructions and results.

One of the main constructions for constant dimension codes is based on rank-metric codes. For two $k \times \ell$ matrices $A$ and $B$ over $\mathbb{F}_q$ the *rank distance* is defined by

$$d_R(A,B) \stackrel{\text{def}}{=} \text{rank}(A-B) \ .$$

A $[k \times \ell, \varrho, \delta]_q$ *rank-metric code* $\mathcal{C}$ is a linear code, whose codewords are $k \times \ell$ matrices over $\mathbb{F}_q$; they form a linear subspace with dimension $\varrho$ of $\mathbb{F}_q^{k \times \ell}$, and for each two distinct codewords $A$ and $B$ we have that $d_R(A,B) \geq \delta$ (clearly, $\delta \leq \min\{k,\ell\}$). For a $[k \times \ell, \varrho, \delta]_q$ rank-metric code $\mathcal{C}$ it was proved in [6, 10, 14] that

$$\varrho \leq \min\left\{k(\ell-\delta+1), \ell(k-\delta+1)\right\} \ . \tag{1}$$

This bound is attained for all possible parameters and the codes which attain it are called *maximum rank distance* codes (or MRD codes in short).

There is a close connection between constant dimension codes and rank-metric codes [7, 16]. Let $A$ be a $k \times \ell$ matrix over $\mathbb{F}_q$ and let $I_k$ be the $k \times k$ identity matrix. The matrix $[I_k \; A]$ can be viewed as a generator matrix of a $k$-dimensional subspace of $\mathbb{F}_q^{k+\ell}$, and it is called the *lifting* of $A$ [16].

**Example 1.** *Let $A$ and $[I_3 \; A]$ be the following matrices over $\mathbb{F}_2$*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \;\; , \;\; [I_3 \; A] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

*then the subspace obtained by the lifting of $A$ is given by the following* 8 *vectors:*

$$(100110), (010011), (001001), (110101),$$

$$(101111), (011010), (111100), (000000).$$

A constant dimension code $\mathbb{C}$ such that all its codewords are lifted codewords of an MRD code is called a *lifted MRD code* [16]. This code will be denoted by $\mathbb{C}^{\mathrm{MRD}}$. A lifted MRD code constructed from $[k \times (n-k), (n-k)(k-\delta+1), \delta]_q$ MRD code will be called an $(n, k, \delta)_q$ $\mathbb{C}^{\mathrm{MRD}}$.

**Theorem 1.** *[16] If $\mathcal{C}$ is a $[k \times (n-k), (n-k)(k-\delta+1), \delta]_q$ MRD code then $(n, k, \delta)_q$ $\mathbb{C}^{MRD}$ is an $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ code.*

**Remark 1.** *The parameters of the $[k \times (n-k), (n-k)(k-\delta+1), \delta]_q$ MRD code $\mathcal{C}$ in Theorem 1 imply that $k \leq n-k$, by (1).*

Let $\mathbb{V}^{(n,k)}$ be the set of nonzero vectors of $\mathbb{F}_q^n$ whose first $k$ entries form a nonzero vector. The following results were proved in [8].

**Lemma 1.** *The codewords of an $(n, k, \delta)_q$ $\mathbb{C}^{MRD}$ can be partitioned into $q^{(n-k)(k-\delta)}$ sets, called parallel classes, each one of size $q^{n-k}$, such that in each parallel class each element of $\mathbb{V}^{(n,k)}$ is contained in exactly one codeword.*

**Corollary 1.** *The codewords of an $(n, k, \delta)_q$ $\mathbb{C}^{MRD}$ can be partitioned into $q^{(n-k)(k-\delta)}$ codes, each one is an $(n, q^{n-k}, 2k, k)_q$ code.*

Our next two definitions is given in this section only for $q = 2$. It will be modified and generalized in Section 4. But, for better understanding and since the next section consider only the case where $q = 2$ we restrict our discussion at this point for this case.

For a given $x \in \mathbb{F}_2^k$, let $\mathbb{V}_x^{(n,k)}$ denote the set nonzero vectors in $\mathbb{F}_2^n$ whose first $k$ entries form the vector $x$. In the sequel, let $\mathbf{0}$ denote the all-zero vector.

A *subspace transversal design* of groupsize $2^{n-k}$, block dimension $k$, and *strength* $t$, denoted by $\mathrm{STD}_2(t, k, n-k)$, is a triple $(\mathbb{V}, \mathbb{G}, \mathbb{B})$, where $\mathbb{V}$ is a set of points, $\mathbb{G}$ is a set of groups, and $\mathbb{B}$ is a set of blocks. These three sets must satisfy the following five properties:

1. $\mathbb{V}$ is a set of size $(2^k - 1)2^{n-k}$ (the *points*). $\mathbb{V}^{(n,k)}$ is used as the set of points $\mathbb{V}$.

2. $\mathbb{G}$ is a partition of $\mathbb{V}$ into $2^k - 1$ classes of size $2^{n-k}$ (the *groups*); the groups which are used are defined by $\mathbb{V}_x^{(n,k)}$, $x \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$.

3. $\mathbb{B}$ is a collection of $k$-dimensional subspaces of $\mathbb{F}_2^n$ which contain nonzero vectors only from $\mathbb{V}^{(n,k)}$ (the *blocks*);

4. each block meets each group in exactly one point;

5. every $t$-dimensional subspace (with points from $\mathbb{V}^{(n,k)}$) which meets each group in at most one point is contained in exactly one block.

An $\mathrm{STD}_2(t,k,m)$ is *resolvable* if the set $\mathbb{B}$ can be partitioned into sets $\mathbb{B}_1,...,\mathbb{B}_s$, where each vector of $\mathbb{V}^{(n,k)}$ is contained in exactly one block of each $\mathbb{B}_i$, $1 \leq i \leq s$. The sets $\mathbb{B}_1,...,\mathbb{B}_s$ are called *parallel classes*. The following theorem was established in [8].

**Theorem 2.** *The codewords of an $(n,k,\delta)_2$ $\mathbb{C}^{MRD}$ form the blocks of a resolvable $STD_2(k-\delta+1,k,n-k)$, with the set of groups $\mathbb{V}_x^{(n,k)}$, $x \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$.*

In the sequel we will represent nonzero elements of the finite field $\mathbb{F}_{2^m}$ in two different ways. The first one is by $m$-tuples over $\mathbb{F}_2$ (in other words, $\mathbb{F}_{2^m}$ is represented by $\mathbb{F}_2^m$) and the second one is by powers of a primitive element $\alpha$ in $\mathbb{F}_{2^m}$. We will not distinguish between these two isomorphic representations. When an $m$-tuple $z$ over $\mathbb{F}_2$ will be multiplied by an element $\beta \in \mathbb{F}_{2^m}$ we will view $z$ as an element in $\mathbb{F}_{2^m}$ and the result will be an element in $\mathbb{F}_{2^m}$ which is also represented by an $m$-tuple over $\mathbb{F}_2$ (an element in $\mathbb{F}_2^m$). Also, when we write $\mathbb{V}_\gamma^{(n,k)}$, where $\gamma \in \mathbb{F}_{2^k}$, it is the same as writing $\mathbb{V}_x^{(n,k)}$, $x \in \mathbb{F}_2^k$, where $x$ is the binary $k$-tuple which represents $\gamma$. Therefore, vectors can be represented by powers of primitive elements in the related finite field. We will use this notation in some cases.

For a set $S \subseteq \mathbb{F}_2^m$ and a nonzero element $\beta \in \mathbb{F}_{2^m}$, we define $\beta S \stackrel{\text{def}}{=} \{\beta x \ : \ x \in S\}$. We note that we can take the set $S$ to be a subspace. The following lemma is a simple observation.

**Lemma 2.** *If $X$ is a $k$-dimensional subspace of $\mathbb{F}_2^m$ and $\beta$ is a nonzero element of $\mathbb{F}_{2^m}$ then $\beta X$ is also a $k$-dimensional subspace of $\mathbb{F}_2^m$.*

# 3 A construction for $q = 2$ and $n = 2k$

Recall that the vectors of $\mathbb{F}_2^{2k} \setminus \{\mathbf{0}\}$ are partitioned into $2^k$ parts, $\mathbb{V}_x^{(2k,k)}$, $x \in \mathbb{F}_2^k$. Let $\mathbb{V}_{\mathbf{0}}$ denote the $k$-dimensional subspace spanned by $\mathbb{V}_{\mathbf{0}}^{(2k,k)}$.

Consider $k$-dimensional subspaces from $\mathcal{G}_2(2k,k)$ of three types:

1. A $k$-dimensional subspace $Y \in \mathcal{G}_2(2k,k)$ is of Type A if for each $x \in \mathbb{F}_2^k \setminus \{\mathbf{0}\}$, $Y$ contains exactly one vector from $\mathbb{V}_x^{(2k,k)}$, and $Y$ does not contain any vector from $\mathbb{V}_{\mathbf{0}}^{(2k,k)}$.

2. A $k$-dimensional subspace $Y \in \mathcal{G}_2(2k,k)$ is of Type B if $Y$ contains exactly one vector from $\mathbb{V}_{\mathbf{0}}^{(2k,k)}$.

3. A $k$-dimensional subspace $Y \in \mathcal{G}_2(2k,k)$ is of Type C if all the vectors of $Y$ are contained in $\mathbb{V}_{\mathbf{0}}^{(2k,k)}$, i.e. $Y = \mathbb{V}_{\mathbf{0}}$.

One can readily verify that

**Lemma 3.** *If $Z$ is a $k$-dimensional subspace of Type B then $Z$ has the structure*

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(x_0,y_0),(x_0,y_1),(x_1,y_2),(x_1,y_3),\ldots,(x_{2^{k-1}-2},y_{2^k-3}),(x_{2^{k-1}-2},y_{2^k-2})\},$$

*where $\{\mathbf{0},x_0,x_1,\ldots,x_{2^{k-1}-2}\}$ is a $(k-1)$-dimensional subspace of $\mathbb{F}_2^{2k}$ and for each $i$, $0 \leq i \leq 2^{k-1}-2$, we have $z = y_{2i}+y_{2i+1}$.*

For completeness, even so it is not necessary for our discussion, we give the following lemma without a proof.

**Lemma 4.**

- *There exist exactly $2^{k^2}$ distinct $k$-dimensional subspaces of Type A.*
- *There exist exactly $(2^k-1)^2 2^{(k-1)^2}$ distinct $k$-dimensional subspaces of Type B.*
- *There exists exactly one $k$-dimensional subspace of Type C.*

Our construction which follows will yield $2^k-1$ pairwise disjoint spreads in $\mathcal{G}_2(2k,k)$. Each spread will consist of exactly $2^k-1$ subspaces of Type B and exactly two subspaces of Type A. In the construction, a $k$-dimensional subspace $Z$ of $\mathbb{F}_2^{2k}$ will be represented as

$$Z = \{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}\ ,$$

where $x_i, y_i \in \mathbb{F}_2^k$ and $y_i \neq \mathbf{0}$ if $x_i = \mathbf{0}$, $0 \leq i \leq 2^k-2$.

Let $\mathbb{C}_0$ be a $(2k,k,k-1)_2$ $\mathbb{C}^{\text{MRD}}$, i.e. a $(2k,2^{2k},2(k-1),k)_2$ code. By Corollary 1, $\mathbb{C}_0$ can be partitioned into $2^k$ codes, each one is a $(2k,2^k,2k,k)_2$ code. Each one of these $2^k$ codes can be completed to a spread if we add $\mathbb{V}_{\mathbf{0}}$ to the code. $\mathbb{C}_0$ is constructed from a linear rank-metric code $\mathcal{C}$ and therefore one of its codewords is the $k$-dimensional subspace $\{(\mathbf{0},\mathbf{0}),(x_0,\mathbf{0}),(x_1,\mathbf{0}),\ldots,(x_{2^k-2},\mathbf{0})\}$. Since the minimum subspace distance of $\mathbb{C}_0$ is $2(k-1)$, it follows that for each other codeword $\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$ of $\mathbb{C}_0$, at most one of $y_i$'s is the all-zero vector. Therefore,

**Lemma 5.** *The code $\mathbb{C}_0$ can be partitioned into $2^k$ $(2k,2^k,2k,k)_2$ codes, for which, each one which does not contain the codeword $\{(\mathbf{0},\mathbf{0}),(x_0,\mathbf{0}),(x_1,\mathbf{0}),\ldots,(x_{2^k-2},\mathbf{0})\}$, contains exactly $2^k-1$ codewords of the form*

$$\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}\ ,$$

*in which exactly one of the $y_i$'s is the all-zero vector.*

**Corollary 2.** *There exists a $(2k,2^k+1,2k,k)_2$ code which contains $\mathbb{V}_{\mathbf{0}}$ as a codeword and for each codeword $\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$ at most one of the $y_i$'s is the all-zero vector.*

Let $\mathbb{C}$ be a $(2k,2^k+1,2k,k)_2$ code as described in Corollary 2, i.e. it contains $\mathbb{V}_{\mathbf{0}}$ as a codeword and for each codeword $\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$ at most one of the $y_i$'s is the all-zero vector. Let $\overleftrightarrow{\mathbb{C}}$ be the $(2k,2^k+1,2k,k)_2$ code obtained from $\mathbb{C}$ as follows

$$\overleftrightarrow{\mathbb{C}} \overset{\text{def}}{=} \Big\{\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}\ :\ \{(\mathbf{0},\mathbf{0}),(y_0,x_0),(y_1,x_1),\ldots,(y_{2^k-2},x_{2^k-2})\} \in \mathbb{C}\Big\}.$$

Henceforth, let $\alpha$ be a primitive element in the field $\mathbb{F}_{2^k}$. As a consequence of Lemma 5 and Corollary 2 we have

**Lemma 6.** *The code $\overleftrightarrow{\mathbb{C}}$ is a spread, in $\mathcal{G}_2(2k,k)$, which consists of exactly $2^k-1$ subspaces of Type B and exactly two subspaces of Type A. One of the two subspaces of Type A has the form* $\{(\mathbf{0},\mathbf{0}),(x_0,\mathbf{0}),(x_1,\mathbf{0}),\ldots,(x_{2^k-2},\mathbf{0})\}$.

Now, we are in a position to define $2^k-1$ pairwise disjoint spreads in $\mathcal{G}_2(2k,k)$. For our first spread $\mathbb{S}_0$ defined as follows, we distinguish between two cases:
**Case 1:** If there is no subspace in $\overleftrightarrow{\mathbb{C}}$ of the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$, for any $j$, $0\le j\le 2^k-2$, then

$$\mathbb{S}_0 \stackrel{\text{def}}{=} \Big\{\{(\mathbf{0},\mathbf{0}),(\alpha^0,y_0+\alpha^0),(\alpha^1,y_1+\alpha^1),(\alpha^2,y_2+\alpha^2),\ldots,(\alpha^{2^k-2},y_{2^k-2}+\alpha^{2^k-2})\} \;:$$

$$\{(\mathbf{0},\mathbf{0}),(\alpha^0,y_0),(\alpha^1,y_1),(\alpha^2,y_2),\ldots,(\alpha^{2^k-2},y_{2^k-2})\}\in\overleftrightarrow{\mathbb{C}}\Big\}$$

$$\cup\Big\{\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0+\alpha^{i_0}),(\alpha^{i_0},y_1+\alpha^{i_0}),\ldots,(\alpha^{i_{2^{k-1}-2}},y_{2^k-4}+\alpha^{i_{2^{k-1}-2}}),(\alpha^{i_{2^{k-1}-2}},y_{2^k-3}+\alpha^{i_{2^{k-1}-2}})\} \;:$$

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0),(\alpha^{i_0},y_1),\ldots,(\alpha^{i_{2^{k-1}-2}},y_{2^k-4}),(\alpha^{i_{2^{k-1}-2}},y_{2^k-3})\}\in\overleftrightarrow{\mathbb{C}}\Big\}\,.$$

**Case 2:** If there exists a subspace in $\overleftrightarrow{\mathbb{C}}$ of the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$ for some $j$, $0\le j\le 2^k-2$, then

$$\mathbb{S}_0 \stackrel{\text{def}}{=} \Big\{\{(\mathbf{0},\mathbf{0}),(\alpha^0,y_0+\alpha^0),(\alpha^1,y_1+\alpha^2),(\alpha^2,y_2+\alpha^4),\ldots,(\alpha^{2^k-2},y_{2^k-2}+\alpha^{2^k-3})\} \;:$$

$$\{(\mathbf{0},\mathbf{0}),(\alpha^0,y_0),(\alpha^1,y_1),(\alpha^2,y_2),\ldots,(\alpha^{2^k-2},y_{2^k-2})\}\in\overleftrightarrow{\mathbb{C}}\Big\}$$

$$\cup\Big\{\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0+\alpha^{2\cdot i_0}),(\alpha^{i_0},y_1+\alpha^{2\cdot i_0}),\ldots,(\alpha^{i_{2^{k-1}-2}},y_{2^k-4}+\alpha^{2\cdot i_{2^{k-1}-2}}),(\alpha^{i_{2^{k-1}-2}},y_{2^k-3}+\alpha^{2\cdot i_{2^{k-1}-2}})\} \;:$$

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0),(\alpha^{i_0},y_1),\ldots,(\alpha^{i_{2^{k-1}-2}},y_{2^k-4}),(\alpha^{i_{2^{k-1}-2}},y_{2^k-3})\}\in\overleftrightarrow{\mathbb{C}}\Big\}\,.$$

The following two lemmas can be easily verified.

**Lemma 7.** *If* $\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0),(\alpha^{i_0},y_1),\ldots,(\alpha^{i_{2^{\ell-1}-2}},y_{2^\ell-4}),(\alpha^{i_{2^{\ell-1}-2}},y_{2^\ell-3})\}$ *is an $\ell$-dimensional subspace and* $\{(\mathbf{0},\mathbf{0}),(\alpha^{i_0},v_0),\ldots,(\alpha^{i_{2^{\ell-1}-2}},v_{2^{\ell-1}-2})\}$ *is an $(\ell-1)$-dimensional subspace then*

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(\alpha^{i_0},y_0+v_0),(\alpha^{i_0},y_1+v_0),\ldots,(\alpha^{i_{2^{\ell-1}-2}},y_{2^\ell-4}+v_{2^{\ell-1}-2}),(\alpha^{i_{2^{\ell-1}-2}},y_{2^\ell-3}+v_{2^{\ell-1}-2})\}$$

*is an $\ell$-dimensional subspace.*

**Lemma 8.** *If* $\{(\mathbf{0},\mathbf{0}),(\alpha^{i_0},y_0),(\alpha^{i_1},y_1),\ldots,(\alpha^{i_{2^\ell-2}},y_{2^\ell-2})\}$ *and* $\{(\mathbf{0},\mathbf{0}),(\alpha^{i_0},v_0),(\alpha^{i_1},v_1),\ldots,(\alpha^{i_{2^\ell-2}},v_{2^\ell-2})\}$ *are two distinct $\ell$-dimensional subspaces then*

$$\{(\mathbf{0},\mathbf{0}),(\alpha^{i_0},y_0+v_0),(\alpha^{i_1},y_1+v_1),\ldots,(\alpha^{i_{2^\ell-2}},y_{2^\ell-2}+v_{2^\ell-2})\}$$

*is an $\ell$-dimensional subspace.*

**Lemma 9.** $\mathbb{S}_0$ *is a spread.*

*Proof.* By Lemma 6, $\overleftrightarrow{\mathbb{C}}$ is a spread. By Lemmas 7 and 8, the elements defined in $\mathbb{S}_0$ are $k$-dimensional subspaces. It is easy to verify by the definition of $\mathbb{S}_0$ that if $X$ and $Y$ are two disjoint $k$-dimensional subspaces of $\overleftrightarrow{\mathbb{C}}$ then their related $k$-dimensional subspaces $X'$ and $Y'$, respectively (constructed from $X$ and $Y$, respectively) in $\mathbb{S}_0$ are also disjoint. Therefore, $\mathbb{S}_0$ is a spread. □

By Lemma 6 and by the definition of $\mathbb{S}_0$ we have that

**Lemma 10.** *The spread $\mathbb{S}_0$ consists of exactly $2^k - 1$ subspaces of Type B and exactly two subspaces of Type A.*

By Lemma 6 and by the definition of $\mathbb{S}_0$ we also have that

**Lemma 11.** *No subspace in $\mathbb{S}_0$ has the form $\{(\mathbf{0},\mathbf{0}),(x_0,\mathbf{0}),(x_1,\mathbf{0}),\ldots,(x_{2^k-2},\mathbf{0})\}$. At most one of the subspaces of Type A in $\mathbb{S}_0$ has the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$, for some $j$, $0 \le j \le 2^k-2$.*

**Lemma 12.** *Let $X_1, X_2, \ldots, X_{2^\ell-1}$ be $2^\ell-1$ $(\ell-1)$-dimensional subspaces of $F_2^\ell$. If each nonzero element of $\mathbb{F}_2^\ell$ is contained in exactly $2^{\ell-1}-1$ of these $2^\ell-1$ subspaces then $X_1, X_2, \ldots, X_{2^\ell-1}$ are distinct $(\ell-1)$-dimensional subspaces.*

*Proof.* First note the $\dim(X_i \cap X_j) = \ell - 2$ for $1 \le i < j \le 2^\ell - 1$. Hence, for any given $r$, $1 \le r \le 2^\ell - 1$,

$$\sum_{i=1}^{2^\ell-1} |X_i \cap X_r| = 2^\ell - 1 + \lambda_r(2^{\ell-1}-1) + (2^\ell - 1 - \lambda_r)(2^{\ell-2}-1)\ , \tag{2}$$

where $\lambda_r$ is the number of subspaces in $X_1, X_2, \ldots, X_{2^\ell-1}$ which equals $X_r$. On the other hand, since each nonzero element of $X_r$ is contained in exactly $2^{\ell-1}-1$ of these $2^\ell-1$ subspaces then

$$\sum_{i=1}^{2^\ell-1} |X_i \cap X_r| = 2^\ell - 1 + (2^{\ell-1}-1)(2^{\ell-1}-1)\ . \tag{3}$$

The solution for the equations (2) and (3) is $\lambda_r = 1$ which proves the lemma. □

**Corollary 3.** *Let*

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z_1),(x_0,y_0),(x_0,y_1),(x_1,y_2),(x_1,y_3),\ldots,(x_{2^{k-1}-2},y_{2^k-4}),(x_{2^{k-1}-2},y_{2^k-3})\}$$

*and*

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z_2),(u_0,v_0),(u_0,v_1),(u_1,v_2),(u_1,v_3),\ldots,(u_{2^{k-1}-2},v_{2^k-4}),(u_{2^{k-1}-2},v_{2^k-3})\},$$

*be two subspaces of Type B in $\mathbb{S}_0$. Then, the $(k-1)$-dimensional subspaces $\{\mathbf{0},x_0,x_1,\ldots,x_{2^{k-1}-2}\}$ and $\{\mathbf{0},u_0,u_1,\ldots,u_{2^{k-1}-2}\}$ are not equal.*

Given the spread $\mathbb{S}_0$, we define the spread $\mathbb{S}_i$, $1 \le i \le 2^k-2$ as follows.

$$\mathbb{S}_i \overset{\text{def}}{=} \left\{ \{(\mathbf{0},\mathbf{0}),(x_0,\alpha^i y_0),(x_1,\alpha^i y_1),\ldots,(x_{2^k-2},\alpha^i y_{2^k-2})\} \ : \ \{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\} \in \mathbb{S}_0 \right\}.$$

**Lemma 13.** *For each $i$, $1 \le i \le 2^k - 2$, $\mathbb{S}_i$ is a spread.*

*Proof.* Follows immediately from the following two simple observations. The first one is that if $\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$ is a $k$-dimensional subspace then also the set $\{(\mathbf{0},\mathbf{0}),(x_0,\alpha^i y_0),(x_1,\alpha^i y_1),\ldots,(x_{2^k-2},\alpha^i y_{2^k-2})\}$ is a $k$-dimensional subspace. The second one is that if the set $\mathcal{F} \overset{\text{def}}{=} \{(u_j,v_j) \ : \ u_j,v_j \in \mathbb{F}_2^k,\ (u_j,v_j) \neq (\mathbf{0},\mathbf{0}),\ 0 \le j \le 2^{2k}-2\}$ contains all the $2^{2k}-1$ nonzero elements of $\mathbb{F}_2^k \times \mathbb{F}_2^k$ then the set $\{(u_j,\alpha^i v_j) \ : \ (u_j,v_j) \in \mathcal{F},\ 0 \le j \le 2^{2k}-2\}$ also contains all the $2^{2k}-1$ nonzero elements of $\mathbb{F}_2^k \times \mathbb{F}_2^k$. □

It is easily verified that

**Lemma 14.** *For each $0 \le i \le 2^k - 2$, if the $k$-dimensional subspace*

$$\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$$

*is of Type A (Type B, respectively) then the $k$-dimensional subspace*

$$\{(\mathbf{0},\mathbf{0}),(x_0,\alpha^i y_0),(x_1,\alpha^i y_1),\ldots,(x_{2^k-2},\alpha^i y_{2^k-2})\}$$

*is also of Type A (Type B, respectively).*

**Lemma 15.** *For each $i_1$, $i_2$, such that $0 \le i_1 < i_2 \le 2^k-2$, the spreads $\mathbb{S}_{i_1}$ and $\mathbb{S}_{i_2}$ are disjoint.*

*Proof.* By the definition of Type A and Type B, and by the definition of $\mathbb{S}_j$, we have that for each $j$, $1 \le j \le 2^k-2$, the number of subspaces of Type A (Type B, respectively) in $\mathbb{S}_0$ is equal to the number of subspaces of Type A (Type B, respectively) in $\mathbb{S}_j$. Therefore, by Lemma 10, in $\mathbb{S}_j$, $1 \le j \le 2^k-2$, there are exactly $2^k-1$ subspaces of Type B and exactly two subspaces of Type A. We distinguish now between the two types of subspaces.
**Case 1:** Subspaces of Type B.

By Corollary 3, if $\{\mathbf{0},x_0,x_1,\ldots,x_{2^{k-1}-2}\}$ is a $(k-1)$-dimensional subspaces of $\mathbb{F}_2^k$ then there is at most one subspace of the form

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},z),(x_0,y_0),(x_0,y_1),(x_1,y_2),(x_1,y_3),\ldots,(x_{2^{k-1}-2},y_{2^k-4}),(x_{2^{k-1}-2},y_{2^k-3})\},$$

in $\mathbb{S}_0$. By the construction of spread $\mathbb{S}_j$, $1 \le j \le 2^k-2$, from $\mathbb{S}_0$, we have that the spreads $\mathbb{S}_{i_1}$ and $\mathbb{S}_{i_2}$ can have a subspace of Type B in common if for such a $(k-1)$-dimensional subspace $\{\mathbf{0},x_0,x_1,\ldots,x_{2^{k-1}-2}\}$, the two $k$-dimensional subspaces

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\alpha^{i_1}z),(x_0,\alpha^{i_1}y_0),(x_0,\alpha^{i_1}y_1),(x_1,\alpha^{i_1}y_2),(x_1,\alpha^{i_1}y_3),\ldots,(x_{2^{k-1}-2},\alpha^{i_1}y_{2^k-4}),(x_{2^{k-1}-2},\alpha^{i_1}y_{2^k-3})\}$$

and

$$\{(\mathbf{0},\mathbf{0}),(\mathbf{0},\alpha^{i_2}z),(x_0,\alpha^{i_2}y_0),(x_0,\alpha^{i_2}y_1),(x_1,\alpha^{i_2}y_2),(x_1,\alpha^{i_2}y_3),\ldots,(x_{2^{k-1}-2},\alpha^{i_2}y_{2^k-4}),(x_{2^{k-1}-2},\alpha^{i_2}y_{2^k-3})\}$$

are equal. This is clearly impossible since $\alpha^{i_1}z \neq \alpha^{i_2}z$. Hence, $\mathbb{S}_{i_1}$ and $\mathbb{S}_{i_2}$ have distinct subspaces of Type B.
**Case 2:** Subspaces of Type A.

A $k$-dimensional subspace of Type A in $\mathbb{S}_0$ has the form

$$\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}\ ,$$

where all the $x_j$'s are different. If $y_r \neq \mathbf{0}$ for some $r$, then

$\{(\mathbf{0},\mathbf{0}),(x_0,\alpha^{i_1}y_0),(x_1,\alpha^{i_1}y_1),\ldots,(x_{2^k-2},\alpha^{i_1}y_{2^k-2})\} \neq \{(\mathbf{0},\mathbf{0}),(x_0,\alpha^{i_2}y_0),(x_1,\alpha^{i_2}y_1),\ldots,(x_{2^k-2},\alpha^{i_2}y_{2^k-2})$ .

By Lemma 11, not all the $y_j$'s are *zeroes* and at most one of the subspaces of Type A in $\mathbb{S}_0$ has the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$, for some $j$, $0 \leq j \leq 2^k-2$. Let

$$\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_{2^k-2},y_{2^k-2})\}$$

and

$$\{(\mathbf{0},\mathbf{0}),(x_0,v_0),(x_1,v_1),\ldots,(x_{2^k-2},v_{2^k-2})\}$$

be the two subspaces of Type A in $\mathbb{S}_0$. Assume a $k$-dimensional subspace of Type A in $\mathbb{S}_{i_1}$ is equal a $k$-dimensional subspace of Type A in $\mathbb{S}_{i_2}$. Then

$$\alpha^{i_1}y_\ell = \alpha^{i_2}v_\ell$$

for each $\ell$, $1 \leq \ell \leq 2^k-2$. It implies that for each $\ell$, $1 \leq \ell \leq 2^k-2$, $\frac{y_\ell}{v_\ell} = \alpha^{i_2-i_1}$, and hence for all $\ell_1$, $\ell_2$, $0 \leq \ell_1 < \ell_2 \leq 2^k-1$ we have $\frac{y_{\ell_2}}{v_{\ell_2}} = \frac{y_{\ell_1}}{v_{\ell_1}}$. We distinguish now between two subcases.

**Case 2.1:** Assume that there is no subspace of the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^j,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$, for any $j$, $0 \leq j \leq 2^k-2$, in $\overleftrightarrow{\mathbb{C}}$. Hence, $\mathbb{S}_0$ contains the $k$-dimensional subspace

$$\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^0),(\alpha^1,\alpha^1),\ldots,(\alpha^{2^k-2},\alpha^{2^k-2})\}$$

and the second subspace of type A does not has the form

$$\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\},$$

for any $j$, $1 \leq j \leq 2^k-2$. It implies that there exist $\ell_1$, $\ell_2$, $0 \leq \ell_1 < \ell_2 \leq 2^k-1$ such that $\frac{y_{\ell_2}}{v_{\ell_2}} \neq \frac{y_{\ell_1}}{v_{\ell_1}}$, a contradiction.

**Case 2.2:** Assume that for some $j$, $0 \leq j \leq 2^k-2$, there exists a subspace of the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j),(\alpha^1,\alpha^{j+1}),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2})\}$, in $\overleftrightarrow{\mathbb{C}}$. Hence, the two subspaces of Type A in $\mathbb{S}_0$ have the form $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^0),(\alpha^1,\alpha^2),(\alpha^2,\alpha^4),\ldots,(\alpha^{2^k-2},\alpha^{2^k-3})\}$ and $\{(\mathbf{0},\mathbf{0}),(\alpha^0,\alpha^j+\alpha^0),(\alpha^1,\alpha^{j+1}+\alpha^2),(\alpha^2,\alpha^{j+2}+\alpha^4),\ldots,(\alpha^{2^k-2},\alpha^{j+2^k-2}+\alpha^{2^k-3})\}$. W.l.o.g. we can assume that $x_0 = \alpha^0$ and $x_1 = \alpha^1$. It implies that $\frac{y_0}{v_0} = \alpha^j+\alpha^0 \neq \alpha^j+\alpha^1 = \frac{y_1}{v_1}$, a contradiction.

Thus, $\mathbb{S}_{i_1}$ and $\mathbb{S}_{i_2}$ are disjoint spreads in $\mathcal{G}_2(2k,k)$. $\square$

**Corollary 4.** *There exists a set of* $2^k-1$ *pairwise disjoint spreads in* $\mathcal{G}_2(2k,k)$.

**Corollary 5.** *There exists a set of* $2^{k+1}-1$ *pairwise disjoint* $k$*-spreads in* $PG(2k+1,2)$.

# 4 A construction for $q > 2$ and $n = 2k$

In this section we will describe a construction of two disjoint spreads in $\mathcal{G}_q(2k,k)$ for any $q > 2$. The idea behind the construction will be similar to the one for $q = 2$. But, since we construct only two disjoint spreads, the analysis will be much simpler. We will start by modifying and generalizing the definition of a subspace transversal design for $q \geq 2$.

For a given $X \in \mathcal{G}_q(k,1)$, let $\mathbb{V}_X^{(n,k)}$ denote the set nonzero vectors in $\mathbb{F}_2^n$ whose first $k$ entries form any given nonzero vector of $X$. Let $\mathbb{V}_{\mathbf{0}}^{(n,k)}$ denote a maximal set of $\frac{q^{n-k}-1}{q-1}$ nonzero vectors in $\mathbb{F}_q^n$ whose first $k$ entries are *zeroes*, for which any two vectors in the set are linearly independent. Let $\mathbb{V}_{\mathbf{0}}$ denote the $k$-dimensional subspace spanned by $\mathbb{V}_{\mathbf{0}}^{(n,k)}$.

A *subspace transversal design* of groupsize $q^{n-k}$, block dimension $k$, and *strength* $t$, denoted by $\mathrm{STD}_q(t,k,n-k)$, is a triple $(\mathbb{V},\mathbb{G},\mathbb{B})$, where $\mathbb{V}$ is a set of points, $\mathbb{G}$ is a set of groups, and $\mathbb{B}$ is a set of blocks. These three sets must satisfy the following five properties:

1. $\mathbb{V}$ is a set of size $\frac{q^k-1}{q-1}q^{n-k}$ (the *points*). $\bigcup_{X\in\mathcal{G}_q(k,1)} \mathbb{V}_X^{(n,k)}$ is used as the set of points $\mathbb{V}$.
2. $\mathbb{G}$ is a partition of $\mathbb{V}$ into $\frac{q^k-1}{q-1}$ classes of size $q^{n-k}$ (the *groups*); the groups which are used are defined by $\mathbb{V}_X^{(n,k)}$, $X \in \mathcal{G}_q(k,1)$.
3. $\mathbb{B}$ is a collection of $k$-dimensional subspaces of $\mathbb{F}_q^n$ which contain nonzero vectors only from $\mathbb{V}^{(n,k)}$ (the *blocks*);
4. each block meets each group in exactly one point;
5. every $t$-dimensional subspace (with points from $\mathbb{V}$) which meets each group in at most one point is contained in exactly one block.

An $\mathrm{STD}_q(t,k,m)$ is *resolvable* if the set $\mathbb{B}$ can be partitioned into sets $\mathbb{B}_1,...,\mathbb{B}_s$, where each vector of $\mathbb{V}^{(n,k)}$ is contained in exactly one block of each $\mathbb{B}_i$, $1 \le i \le s$. The sets $\mathbb{B}_1,...,\mathbb{B}_s$ are called *parallel classes*. The following theorem was established in [8].

**Theorem 3.** *The codewords of an $(n,k,\delta)_q$ $\mathbb{C}^{MRD}$ form the blocks of a resolvable $STD_q(k-\delta+1,k,n-k)$, with the set of groups $\mathbb{V}_X^{(n,k)}$, $X \in \mathcal{G}_q(k,1)$.*

We consider $k$-dimensional subspaces of three types:

1. A $k$-dimensional subspace $Y \in \mathcal{G}_q(2k,k)$ is of Type A if for each $X \in \mathcal{G}_q(k,1)$, $Y$ contains exactly one vector from $\mathbb{V}_X^{(n,k)}$, and $Y$ does not contain any vector from $\mathbb{V}_0^{(n,k)}$.
2. A $k$-dimensional subspace $Y \in \mathcal{G}_q(2k,k)$ is of Type B if $Y$ contains exactly one vector from $\mathbb{V}_0^{(n,k)}$.
3. A $k$-dimensional subspace $Y \in \mathcal{G}_q(2k,k)$ is of Type C if all the vectors of $Y$ are contained in $\mathbb{V}_0^{(n,k)}$.

Throughout this section let $\ell = \frac{q^k-1}{q-1} - 1$. Let $\mathbb{C}_0$ be an $(2k, q^{2k}, 2(k-1), k)_q$ $\mathbb{C}^{\mathrm{MRD}}$. $\mathbb{C}_0$ is constructed from a linear rank-metric code $\mathcal{C}$ and therefore the $k$-dimensional subspace $\langle\{(\mathbf{0},\mathbf{0}),(x_0,0),(x_1,0),\ldots,(x_\ell,0)\}\rangle$ is a codeword of $\mathbb{C}_0$. Since the minimum subspace distance of $\mathbb{C}_0$ is $2(k-1)$, it follows that if $\langle\{(\mathbf{0},\mathbf{0}),(x_0,y_0),(x_1,y_1),\ldots,(x_\ell,y_\ell)\}\rangle$ is another codeword of $\mathbb{C}_0$, then at most one of $y_i$'s is an all-zero vector. Since $|\mathcal{G}_2(k,1)| = \frac{q^k-1}{q-1}$ it follows from Lemma 1 and Corollary 1 that

**Lemma 16.** *The code $\mathbb{C}_0$ has an $(n, q^k, 2k, k)_q$ subcode $\mathbb{C}'_0$ which contains exactly $\frac{q^k-1}{q-1}$ codewords of the form*

$$\langle \{(\mathbf{0}, \mathbf{0}), (x_0, y_0), (x_1, y_1), \ldots, (x_\ell, y_\ell)\} \rangle,$$

*in which exactly one of the $y_i$'s is the all-zero vector.*

**Corollary 6.** *There exists an $(2k, q^k+1, 2k, k)_q$ code which contains $\mathbb{V}_0$ as a codeword and for each codeword $\langle \{(\mathbf{0}, \mathbf{0}), (x_0, y_0), (x_1, y_1), \ldots, (x_{2^k-2}, y_{2^k-2})\} \rangle$ at most one of the $y_i$'s is the all-zero vector.*

Let $\mathbb{C}$ be an $(2k, q^k+1, 2k, k)_q$ code as described in Corollary 6. Let $\overleftrightarrow{\mathbb{C}}$ be the $(2k, q^k+1, 2k, k)_q$ code obtained from $\mathbb{C}$ as follows

$$\overleftrightarrow{\mathbb{C}} \overset{\text{def}}{=} \Big\{ \langle \{(\mathbf{0}, \mathbf{0}), (x_0, y_0), (x_1, y_1), \ldots, (x_\ell, y_\ell)\} \rangle \ : \ \langle \{(\mathbf{0}, \mathbf{0}), (y_0, x_0), (y_1, x_1), \ldots, (y_\ell, x_\ell)\} \rangle \in \mathbb{C} \Big\}.$$

As a consequence of Corollary 6 we have

**Lemma 17.** *The code $\overleftrightarrow{\mathbb{C}}$ is a spread which consists of exactly $\frac{q^k-1}{q-1}$ subspaces of Type B and exactly $q^k + 1 - \frac{q^k-1}{q-1}$ subspaces of Type A.*

**Theorem 4.** *There exist at least two disjoint $(2k, q^k+1, 2k, k)_q$ codes.*

*Proof.* By Corollary 1, $\mathbb{C}_0$ can be partitioned into $q^k$ $(2k, q^k, 2k, k)_q$ codes. Since $q^k > q^k + 1 - \frac{q^k-1}{q-1}$ it follows that at least one of these $q^k$ codes does not contain any of the $q^k + 1 - \frac{q^k-1}{q-1}$ subspaces of Type A which are contained in $\overleftrightarrow{\mathbb{C}}$. Let $\mathbb{C}'$ be this code. $\mathbb{C}' \cup \mathbb{V}_0$ is a $(2k, q^k+1, 2k, k)_q$ code which contains $q^k$ subspaces of Type A and one subspace of Type C. Therefore, $\overleftrightarrow{\mathbb{C}}$ and $\mathbb{C}'$ are disjoint. □

**Corollary 7.** *There exist two disjoint spreads in $\mathcal{G}_q(2k, k)$, $q > 2$.*

**Corollary 8.** *There exist two disjoint $k$-spreads in $PG(2k+1, q)$, $q > 2$.*

# 5 A recursive construction

Let $n = \ell k$, where $\ell \geq 2$. Let $\mathbb{S}_i$, $0 \leq i \leq M-1$, be a set of $M$ pairwise disjoint spreads in $\mathcal{G}_q(n, k)$. We will describe a construction for $M$ pairwise disjoint spreads in $\mathcal{G}_q(n+k, k)$.

First we will define a *partial Grassmannian* $\mathcal{G}_q(n_1, n_2, k)$, $n_1 > n_2 \geq k$, as the set of all $k$-dimensional subspaces from $\mathbb{F}_q^{n_1}$ which are not contained in a given $n_2$-dimensional subspace $U$ of $\mathbb{F}_q^{n_1}$. It can be readily verified that $\mathbb{V}^{(n,k)}$ is a partial Grassmannian $\mathcal{G}_q(n, n-k, k)$, where $\mathbb{V}_0^{(n,k)}$ is the $(n-k)$-dimensional subspace $U$. A spread in $\mathcal{G}_q(n_1, n_2, k)$ is a set $\mathbb{S}$ of pairwise disjoint $k$-dimensional subspaces from $\mathcal{G}_q(n_1, n_2, k)$ such that each nonzero element of $\mathbb{F}_q^{n_1} \setminus U$ is contained in exactly one element of $\mathbb{S}$. A parallelism of $\mathcal{G}_q(n_1, n_2, k)$ is a set of pairwise disjoint spreads in $\mathcal{G}_q(n_1, n_2, k)$ such that each $k$-dimensional subspace of $\mathcal{G}_q(n_1, n_2, k)$ is contained in exactly one of the spreads. Beutelspacher [4] proved that if $k = 2$ then such a parallelism exists

if $n_2 \geq 2$, $n_1 - n_2 = 2^i$, for all $i \geq 1$ and any $q > 2$. If $k = 2$ and $q = 2$ then such a parallelism exists if and only if $n_2 \geq 3$ and $n_1 - n_2$ is even.

In this section we are going to extend this results for $k > 2$. Based on these parallelisms we will present a recursive construction for pairwise disjoint spreads in $\mathcal{G}_q(n,k)$, where $k$ divides $n$ and $n > k$.

Theorem 3 is the key for our constructions. A resolvable $\text{STD}_q(k,k,n-k)$ consists of $q^{(n-k)(k-1)}$ spreads of $\mathbb{V}^{(n,k)}$, i.e. a parallelism in $\mathcal{G}_q(n,n-k,k)$. A resolvable $\text{STD}_q(k,k,n-k)$ is obtained from an $(n,k,1)_q$ $\mathbb{C}^{\text{MRD}}$, which is constructed from a $[k \times (n-k), (n-k)k, 1]_q$ MRD code. Thus, we have

**Theorem 5.** *If $k = n_1 - n_2$ then there exists a parallelism in $\mathcal{G}_q(n_1,n_2,k)$.*

If there exists $M$ pairwise disjoint spreads in $\mathcal{G}_q(n-k,k)$ then they can be combined with $M$ pairwise disjoint spreads in $\mathcal{G}_q(n,n-k,k)$ which exist by Theorem 5 to obtain the following theorem.

**Theorem 6.** *If there exist $M$ pairwise disjoint spreads in $\mathcal{G}_q(n-k,k)$ then there exist $M$ pairwise disjoint spreads in $\mathcal{G}_q(n,k)$.*

*Proof.* By Theorem 5, there exists $M$ pairwise disjoint spreads in $\mathcal{G}_q(n,n-k,k)$, in which the removed $(n-k)$-dimensional subspace is isomorphic to $\mathcal{G}_q(n-k,k)$. Let $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_M$, be these spreads. Let $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_M$ be the $M$ pairwise disjoint spreads in $\mathcal{G}_q(n-k,k)$. Then $\mathbb{S}_1 \cup \mathbb{T}_1, \mathbb{S}_2 \cup \mathbb{T}_2, \ldots, \mathbb{S}_M \cup \mathbb{T}_M$ is a set of $M$ pairwise disjoint spreads in $\mathcal{G}_q(n,k)$. $\square$

**Corollary 9.** *There exists a set of $2^k - 1$ pairwise disjoint spreads in $\mathcal{G}_2(n,k)$ if $n > k$ and $k$ divides $n$.*

**Corollary 10.** *There exist two pairwise disjoint spreads in $\mathcal{G}_q(n,k)$ if $n > k$ and $k$ divides $n$.*

**Corollary 11.** *There exists a set of $2^{k+1} - 1$ pairwise disjoint $k$-spreads in PG($n,2$) if $n > k$ and $k+1$ divides $n+1$.*

**Corollary 12.** *There exist two pairwise disjoint spreads $k$-spreads in PG($n,q$) if $n > k$ and $k+1$ divides $n+1$.*

# 6 Conclusion and problems for future research

Finding $k$-parallelism in $\text{PG}(n,q)$ is an extremely difficult problem. If $k > 1$ then only one such parallelism is known. The goal of this paper was to direct the research for the following slightly easier question. What is the maximum number of pairwise disjoint $k$-spreads in $\text{PG}(n,q)$? This number can be greater than one only if $n > k$ and $k+1$ divides $n+1$ which is the sufficient and necessary condition for the existence of $k$-spreads in $\text{PG}(n,q)$. We proved that two such pairwise disjoint $k$-spreads always exist. If $q = 2$ then we proved the existence of $2^{k+1} - 1$ pairwise disjoint $k$-spreads. We also proved that if $k+1$ divides $n_1 + 1$ and $n_2 + 1$, and $n_1 > n_2 > k$, then there exist a $k$-parallelism in the partial space of dimension $n_1$ from which an $n_2$-subspace was removed. There are many interesting open problems in this topic. We will state them in an increasing order of difficulty by our opinion, from the easiest one to the most difficult one.

1. For any $q > 2$ and $k \geq 1$, improve the lower bounds on the number of pairwise disjoint $k$-spreads in $\mathrm{PG}(n, q)$.
2. For $q = 2$ and any $k > 1$, improve the lower bounds on the number of pairwise disjoint $k$-spreads in $\mathrm{PG}(n, q)$.
3. Find nontrivial necessary conditions for the existence of a parallelism in $\mathcal{G}_q(n_1, n_2, k)$.
4. Find new parameters for which there exists a parallelism in $\mathcal{G}_q(n_1, n_2, k)$.
5. For $q > 2$, find new parameters for which there exists an 1-parallelism in $\mathrm{PG}(n, q)$.
6. For $k > 1$, find new parameters for which there exists a $k$-parallelism in $\mathrm{PG}(n, q)$.
7. For $k > 1$, find an infinite family of $k$-parallelisms in $\mathrm{PG}(n, q)$.

# References


[1] R. D. BAKER, *Partitioning the planes $AG_{2m}(2)$ into 2-designs*, *Discrete Math.*, 15 (1976), 205–211.

[2] R. D. BAKER, J. H. VAN LINT, ND R. M. WILSON, *On the Preparata and Goethals codes*, *IEEE Trans. Inform. Theory*, 29 (1983), 342–345.

[3] A. BEUTELSPACHER, *On parallelisms in finite projective spaces*, *Geometriae Dedicata*, 3 (1974), 35–40.

[4] A. BEUTELSPACHER, *Partial parallelisms in finite projective spaces*, *Geometriae Dedicata*, 36 (1990), 273–278.

[5] A. BEUTELSPACHER AND J. UEBERBERG, *A characteristic property of geometric t-spreads in finite projective spaces, Europ. J. Comb.,* 12 (1991), 277–281.

[6] P. DELSARTE, *Bilinear forms over a finite field, with applications to coding theory*, *Journal of Combinatorial Theory, Series A*, 25 (1978), 226–241.

[7] T. ETZION AND N. SILBERSTEIN, *Error-correcting codes in projective space via rank-metric codes and Ferrers diagrams*, *IEEE Trans. Inform. Theory*, 55 (2009), 2909–2919.

[8] T. ETZION AND N. SILBERSTEIN, *Codes and Designs Related to Lifted MRD Codes, IEEE Trans. Inform. Theory*, 59 (2013), 1004–1017.

[9] T. ETZION AND A. VARDY, *Automorphisms of codes in the Grassmann scheme, arxiv.org/abs/1210.5724*, October 2012.

[10] E. M. GABIDULIN, *Theory of codes with maximum rank distance*, *Problems of Information Transmission*, 21 (1985), 1–12.

[11] N. L. JOHNSON, *Combinatorics of Spreads and Parallelisms*, CRC Press, 2010.

[12] R. Koetter and F. R. Kschischang, *Coding for errors and erasures in random network coding*, *IEEE Trans. Inform. Theory*, 54 (2008), 3579–3591.

[13] G. Lunardon, *Normal spreads*, *Geometriae Dedicata,* 75 (1999), 245–261.

[14] R. M. Roth, *Maximum-rank array codes and their application to crisscross error correction*, *IEEE Trans. Inform. Theory*, 37 (1991), 328–336.

[15] J. Sarmiento, *On point-cyclic resolutions of the* $2-(63,7,15)$ *design associated with PG(5,2)*, *Graphs and Comb.*, 18 (2002), 621–632.

[16] D. Silva, F. R. Kschischang, and R. Koetter, *A rank-metric approach to error control in random network coding*, *IEEE Trans. Inform. Theory*, 54 (2008), 3951–3967.